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Spectral Properties of Non-selfadjoint Difference Operators[1]

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Let L denote the operator generated in $\ell^2(\mathbb{Z})$ by the difference expression $(\ell y)_n = a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1}$, $n \in \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$, where $\{a_n\}_{n\in\mathbb{Z}}$ and $\{b_n\}_{n\in\mathbb{Z}}$ are complex sequences. In this paper we investigated the spectrum, the spectral singularities, and the properties of the principal vectors corresponding to the spectral singularities of L. We also studied similar problems for the discrete Dirac operator M generated in $\ell^2(\mathbb{Z}, \mathbb{C}^2)$ by the system of difference expression

$$(\Lambda y)_n = \begin{pmatrix} (\Lambda_1 y)_n \\ (\Lambda_2 y)_n \end{pmatrix} = \begin{pmatrix} \Delta y_n^{(2)} + p_n y_n^{(1)} \\ -\Delta y_{n-1}^{(1)} + q_n y_n^{(2)} \end{pmatrix}, \qquad n \in \mathbb{Z},$$

where

$$y = \left\{ \begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix} \right\}_{n\in\mathbb{Z}},$$

Δ is the forward difference operator, i.e., $\Delta y_n^{(i)} = y_{n+1}^{(i)} - y_n^{(i)}$, $i = 1, 2$, and $\{p_n\}_{n\in\mathbb{Z}}$, $\{q_n\}_{n\in\mathbb{Z}}$ are complex sequences.

1. INTRODUCTION

Study of the spectral analysis of non-selfadjoint Sturm–Liouville operators (SLO) with continuous and point spectrum was begun by Naimark [13]. He proved that the spectrum of non-selfadjoint SLO consists of the continuous spectrum, the eigenvalues, and the spectral singularities. The spectral singularities are poles of the kernel of the resolvent and are also

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imbedded in the continuous spectrum, but they are not eigenvalues. In [12] the effect of the spectral singularities in the spectral expansion of SLO in terms of the principal vectors was considered. The spectral analysis of the quadratic pencil of Schrödinger, Dirac, and Klein–Gordon operators with spectral singularities was studied in [2–7].

Let L denote the difference operator of second order generated in $\ell^2(\mathbb{Z})$ by

$$(\ell y)_n = a_{n-1} y_{n-1} + b_n y_n + a_n y_{n+1}, \qquad n \in \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\},$$

where $\{a_n\}_{n \in \mathbb{Z}}$, $\{b_n\}_{n \in \mathbb{Z}}$ are complex sequences and $a_n \neq 0$ for all $n \in \mathbb{Z}$. Note that we can write the difference expression $(\ell y)_n$ in the following Sturm–Liouville form,

$$(\ell y)_n = \Delta(a_{n-1} \Delta y_{n-1}) + h_n y_n,$$

where $h_n = a_{n-1} + a_n + b_n$ and Δ is the forward difference operator. The purpose of this paper is to extend some of the results of papers [3, 7] to the operator L; i.e., we investigate the spectrum, the spectral singularities, and the properties of the principal vectors corresponding to the spectral singularities of L. We also study similar problems for the non-selfadjoint discrete Dirac operator M generated in $\ell^2(\mathbb{Z}, \mathbb{C}^2)$ by the system of difference expression

$$(\Lambda y)_n = \begin{pmatrix} (\Lambda_1 y)_n \\ (\Lambda_2 y)_n \end{pmatrix} = \begin{pmatrix} \Delta y_n^{(2)} + p_n y_n^{(1)} \\ -\Delta y_{n-1}^{(1)} + q_n y_n^{(2)} \end{pmatrix}, \qquad n \in \mathbb{Z},$$

where $\{p_n\}_{n \in \mathbb{Z}}$ and $\{q_n\}_{n \in \mathbb{Z}}$ are complex sequences.

Note that the spectral analysis of the selfadjoint difference equations and operators have been treated by various authors (for the relevant references one may consult Agarwal and Wong [1] or Berezanski [8]).

2. JOST SOLUTIONS OF $(\ell y)_n = \lambda y_n$

Let us consider the difference equation

$$a_{n-1} y_{n-1} + b_n y_n + a_n y_{n-1} = \lambda y_n, \qquad n \in \mathbb{Z}, \tag{2.1}$$

where λ is a spectral parameter. Suppose that the complex sequences $\{a_n\}_{n \in \mathbb{Z}}$ and $\{b_n\}_{n \in \mathbb{Z}}$ satisfy

$$\sum_{n \in \mathbb{Z}} |n| \big(|1 - a_n| + |b_n| \big) < \infty. \tag{2.2}$$

The following result is obtained in [11]: Under condition (2.2), Eq. (2.1) has unique solutions

$$e_n^+(z) = \alpha_n^+ e^{inz}\left(1 + \sum_{m=1}^{\infty} A_{n,m}^+ e^{imz}\right), \qquad n \in \mathbb{Z}, \tag{2.3}$$

$$e_n^-(z) = \alpha_n^- e^{-inz}\left(1 + \sum_{-\infty}^{m=-1} A_{n,m}^- e^{-imz}\right), \qquad n \in \mathbb{Z}, \tag{2.4}$$

for $\lambda = 2\cos z$, where $z \in \overline{\mathbb{C}}_+ := \{z\colon\ z \in \mathbb{C},\ \operatorname{Im} z \geq 0\}$, and $A_{n,m}^{\pm}$, $\alpha_n^{\pm}$ are expressed in terms of $\{a_n\}_{n\in\mathbb{Z}}$ and $\{b_n\}_{n\in\mathbb{Z}}$ as

$$\alpha_n^+ = \left\{\prod_{k=n}^{\infty} a_k\right\}^{-1}, \qquad A_{n,1}^+ = -\sum_{k=n+1}^{\infty} b_k,$$

$$A_{n,2}^+ = \sum_{k=n+1}^{\infty}\left\{\left(1 - a_k^2\right) + b_k \sum_{s=k+1}^{\infty} b_s\right\},$$

$$A_{n,m+2}^+ = A_{n+1,m}^+ + \sum_{k=n+1}^{\infty}\left\{\left(1 - a_k^2\right)A_{k+1,m}^+ - b_k A_{k,m+1}^+\right\},$$

$$m = 1,2,\ldots;\ n \in \mathbb{Z},$$

$$\alpha_n^- = \left\{\prod_{-\infty}^{k=n-1} a_k\right\}^{-1}, \qquad A_{n,-1}^- = -\sum_{-\infty}^{k=n-1} b_k,$$

$$A_{n,-2}^- = \sum_{-\infty}^{k=n-1}\left(a_k^2 - 1\right) - \sum_{-\infty}^{k=n} b_k \sum_{-\infty}^{s=k} b_s,$$

$$A_{n+1,m-2}^- = A_{n,m}^- + \sum_{-\infty}^{k=n-1}\left(1 - a_k^2\right)A_{k,m}^- + \sum_{-\infty}^{k=n} b_k A_{k,m-1}^-,$$

$$m = -1,-2,\ldots;\ n \in \mathbb{Z}.$$

Moreover $A_{n,m}^{\pm}$ satisfy

$$|A_{n,m}^+| \leq c \sum_{k=n+[\frac{m}{2}]}^{\infty}\left(|1 - a_k| + |b_k|\right), \tag{2.5}$$

$$|A_{n,m}^-| \leq c \sum_{-\infty}^{k=n+[\frac{m}{2}]+1}\left(|1 - a_k| + |b_k|\right),$$

where $[\frac{m}{2}]$ is the integer part of $\frac{m}{2}$ and $c > 0$ is a constant. Therefore

$e_n^+(z)$ and $e_n^-(z)$ $(n \in \mathbb{Z})$ are analytic with respect to z in $\mathbb{C}_+ = \{z\colon z \in \mathbb{C}, \operatorname{Im} z > 0\}$ and continuous in $\overline{\mathbb{C}}_+$, and

$$\lim_{n\to\infty} e_n^+(z)e^{-inz} = 1, \qquad \lim_{n\to-\infty} e_n^-(z)e^{inz} = 1,\ z \in \overline{\mathbb{C}}_+, \tag{2.6}$$

$$e_n^{\pm}(z) = \alpha_n^{\pm} e^{\pm inz}[1 + o(1)], \qquad n \in \mathbb{Z},\ z = \xi + i\tau,\ \tau \to \infty \tag{2.7}$$

hold. Analogously to the Sturm–Liouville equation the solutions $e^+(z) := \{e_n^+(z)\}_{n\in\mathbb{Z}}$ and $e^-(z) := \{e_n^-(z)\}_{n\in\mathbb{Z}}$ are called Jost solutions of (2.1).

The Wronskian of two solutions $y = \{y_n(\lambda)\}_{n\in\mathbb{Z}}$ and $u = \{u_n(\lambda)\}_{n\in\mathbb{Z}}$ of (2.1) is defined by

$$W[y,u] = a_n[y_n(\lambda)u_{n+1}(\lambda) - y_{n+1}(\lambda)u_n(\lambda)].$$

Hereafter, we will denote by ξ the real parameters. It is obvious that $e^+(-\xi) := \{e_n^+(-\xi)\}_{n\in\mathbb{Z}}$ and $e^-(-\xi) := \{e_n^-(-\xi)\}_{n\in\mathbb{Z}}$ are also solutions of (2.1) for $\lambda \in 2\cos\xi$. Using (2.6) we get that

$$W[e^{\pm}(\xi), e^{\pm}(-\xi)] = \mp 2i \sin\xi.$$

So the pairs $\{e_n^+(\xi)\}_{n\in\mathbb{Z}}, \{e_n^+(-\xi)\}_{n\in\mathbb{Z}}$, and $\{e_n^-(\xi)\}_{n\in\mathbb{Z}}, \{e_n^-(-\xi)\}_{n\in\mathbb{Z}}$ form two fundamental systems of solutions of (2.1) for $\lambda = 2\cos\xi$ and $\xi \neq k\pi$, $k \in \mathbb{Z}$. We have the relation

$$e_n^+(\xi) = \gamma(\xi)e_n^-(\xi) + \eta(\xi)e_n^-(-\xi), \qquad n \in \mathbb{Z},$$

for $\xi \neq k\pi$, $k \in \mathbb{Z}$, where

$$\begin{aligned} \eta(\xi) &= -\frac{W[e^+(\xi), e^-(\xi)]}{2i\sin\xi}, \\ \gamma(\xi) &= \frac{W[e^+(\xi), e^-(-\xi)]}{2i\sin\xi}. \end{aligned} \tag{2.8}$$

Moreover the function η has an analytic continuation to the half-plane $\mathbb{C}_+$.

3. CONTINUOUS SPECTRUM OF L

THEOREM 3.1. *If* (2.2) *holds, then* $\sigma_c(L) = [-2, 2]$, *where* $\sigma_c(L)$ *denotes the continuous spectrum of* L.

Proof. Let L_1 and L_2 denote the operators generated in $\ell^2(\mathbb{Z})$ by the difference expressions $(\ell_1 y)_n = y_{n-1} + y_{n+1}$, $(\ell_2 y)_n = (a_{n-1} - 1)y_{n-1} + (a_n - 1)y_{n+1} + b_n y_n$, $n \in \mathbb{Z}$, respectively. It is evident that

$$L = L_1 + L_2, \qquad L_1 = L_1^*,$$

and

$$\sigma(L_1) = \sigma_c(L_1) = [-2, 2],$$

where $\sigma(L_1)$ denote the spectrum of L_1 [11]. From (2.2) we find that L_2 is a compact operator in $\ell^2(\mathbb{Z})$. Using the Weyl theorem of a compact perturbation we obtain

$$\sigma_c(L) = \sigma_c(L_1) = [-2, 2]$$

[10, p. 13]. ∎

4. EIGENVALUES AND SPECTRAL SINGULARITIES OF L

It is known that the function η has an analytic continuation from the real axis to $\mathbb{C}_+$ (see (2.8)). If we define

$$f(z) = -2i\eta(z)\sin z, \qquad z \in \overline{\mathbb{C}}_+,$$

then

$$f(z) = W[e^+(z), e^-(z)], \qquad z \in \overline{\mathbb{C}}_+ \tag{4.1}$$

by (2.8). So we get that f is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}}_+$, and

$$f(z) = f(z + 2\pi).$$

Let us define the semi-strip $P_0 = \{z\colon z = \xi + i\tau,\ -\frac{\pi}{2} \le \xi \le \frac{3\pi}{2},\ \tau > 0\}$ and $P = P_0 \cup [-\frac{\pi}{2}, \frac{3\pi}{2}]$.

We will denote the set of all eigenvalues of L by $\sigma_d(L)$. From the definition of the eigenvalues of L we have

$$\sigma_d(L) = \{\lambda\colon \lambda = 2\cos z,\ z \in P_0,\ f(z) = 0\}. \tag{4.2}$$

For all $z \in P$ and $f(z) \neq 0$, we define

$$G_{n,m}(z) = \begin{cases} \dfrac{e_n^+(z)e_m^-(z)}{f(z)}, & m = n-1, n-2, \ldots, \\[2ex] \dfrac{e_m^+(z)e_n^-(z)}{f(z)}, & m = n, n+1, \ldots. \end{cases} \tag{4.3}$$

The function $G_{n,m}(z)$ is called the Green's function of L. It is obvious that

$$\begin{aligned} (R(L)\varphi)_n &:= \sum_{m\in\mathbb{Z}} G_{n,m}(z)\varphi_m, \\ &\qquad \varphi = \{\varphi_m\}_{m\in\mathbb{Z}} \in \ell^2(\mathbb{Z}),\ n \in \mathbb{Z}, \end{aligned} \tag{4.4}$$

is the resolvent of L.

If $f(z_0) = 0$, for some $z_0 \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$, then $\lambda_0 = 2\cos z_0 \in [-2, 2] = \sigma_c(L)$. So from (4.2)–(4.4) we get that

$$\sigma_{ss}(L) = \left\{\lambda: \lambda = 2\cos z, z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right], f(z) = 0\right\}, \tag{4.5}$$

where the $\sigma_{ss}(L)$ denote the set of all spectral singularities of L [14, p. 306].

THEOREM 4.1. *Under condition* (2.2)

(i) *The set of eigenvalues of L is bounded and countable, and its limit points lie only in* $[-2, 2]$.

(ii) $\sigma_{ss}(L) \subset [-2, 2]$, $\sigma_{ss}(L) = \overline{\sigma_{ss}(L)}$, *and* $\mu\{\sigma_{ss}(L)\} = 0$, *where* $\mu\{\sigma_{ss}(L)\}$ *denotes the linear Lebesgue measure of* $\sigma_{ss}(L)$.

Proof. In order to investigate the quantitative properties of the eigenvalues and the spectral singularities of L, by (4.2) and (4.5), we need to discuss the quantitative properties of the zeros of f in P.

Using (2.3), (2.4), and (4.1) we have

$$\begin{aligned} f(z) &= W[e^+(z), e^-(z)] = a_0[e_0^+(z)e_1^-(z) - e_1^+(z)e_0^-(z)] \\ &= a_0\left[\alpha_0^+\alpha_1^- e^{-iz}\left(1 + \sum_{m=1}^{\infty} A_{0,m}^+ e^{imz}\right)\left(1 + \sum_{-\infty}^{m=-1} A_{1,m}^- e^{-imz}\right)\right. \\ &\qquad \left. - \alpha_1^+\alpha_0^- e^{iz}\left(1 + \sum_{m=1}^{\infty} A_{1,m}^+ e^{imz}\right)\left(1 + \sum_{-\infty}^{m=-1} A_{0,m}^- e^{-imz}\right)\right] \\ &= \left(\prod_{k\in\mathbb{Z}} a_k\right)^{-1}\left[e^{-iz}\left(1 + \sum_{m=1}^{\infty} A_{0,m}^+ e^{imz}\right)\left(1 + \sum_{-\infty}^{m=-1} A_{1,m}^- e^{-imz}\right)\right. \\ &\qquad \left. - a_0^2 e^{iz}\left(1 + \sum_{m=1}^{\infty} A_{1,m}^+ e^{imz}\right)\left(1 + \sum_{-\infty}^{m=-1} A_{0,m}^- e^{-imz}\right)\right]. \end{aligned} \tag{4.6}$$

Consequently we get, by (2.5) and (4.6), that

$$f(z) = \left(\prod_{k\in\mathbb{Z}} a_k\right)^{-1} e^{-iz}[1 + o(1)], \qquad z \in P_0,\ z = \xi + i\tau,\ \tau \to \infty. \tag{4.7}$$

Now (4.7) shows the boundedness of the zeros of f in P_0. Since f is a 2π periodic function and is analytic in $\mathbb{C}_+$, consequently we obtain that f has at most a countable number of zeros in P_0. By the uniqueness of analytic functions we find that the limit points of zeros of f in P_0 can lie only in $[-\frac{\pi}{2}, \frac{3\pi}{2}]$. The closedness and the property of having zero linear Lebesgue measure of the set of zeros of f lying in $[-\frac{\pi}{2}, \frac{3\pi}{2}]$ can be obtained from the boundary uniqueness theorem of analytic functions [9]. ∎

DEFINITION 4.1. The multiplicity of a zero of f in P is called the multiplicity of the corresponding eigenvalue or spectral singularity of L.

THEOREM 4.2. *If, for some* $\varepsilon > 0$,

$$\sup_{n \in \mathbb{Z}} \left\{ e^{\varepsilon |n|} \left(|1 - a_n| + |b_n| \right) \right\} < \infty \tag{4.8}$$

holds, then the operator L has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.

Proof. From (2.5) and (4.8) we find that

$$|A_{j,m}^{+}| \le c e^{-\frac{\varepsilon}{4} m}, \qquad j = 0, 1;\ m = 1, 2, \dots, \tag{4.9}$$

$$|A_{j,m}^{-}| \le c e^{-\frac{\varepsilon}{4} m}, \qquad j = 0, 1;\ m = -1, -2, \dots, \tag{4.10}$$

where $c > 0$ is a constant. By (4.6), (4.9), and (4.10) we observe that the function f has an analytic continuation to the half-plane $\operatorname{Im} z > -\frac{\varepsilon}{4}$. Since f is a 2π periodic function, the limit points of its zeros in P cannot lie in $[-\frac{\pi}{2}, \frac{3\pi}{2}]$. Using Theorem 4.1 we get that the bounded sets $\sigma_d(L)$ and $\sigma_{ss}(L)$ have no limit points, i.e., the sets $\sigma_d(L)$ and $\sigma_{ss}(L)$ have a finite number of elements. From analyticity of f in $\operatorname{Im} z > -\frac{\varepsilon}{4}$ we obtain that all zeros of f in P have a finite multiplicity. Consequently all eigenvalues and spectral singularities have a finite multiplicity. ∎

Since $\lambda = 2\cos z$ maps the semi-strip P_0 to the domain $\Omega := \mathbb{C} \setminus [-2, 2]$, under condition (2.2), the functions

$$E_n^{+}(\lambda) := e_n^{+}\left(\arccos\frac{\lambda}{2}\right), \qquad n \in \mathbb{Z},$$

$$E_n^{-}(\lambda) := e_n^{-}\left(\arccos\frac{\lambda}{2}\right), \qquad n \in \mathbb{Z},$$

$$F(\lambda) := f\left(\arccos\frac{\lambda}{2}\right),$$

are analytic in Ω and continuous up to the boundary of Ω (i.e., up to the interval $[-2, 2]$). It is obvious that $E^{+}(\lambda) = \{E_n^{+}(\lambda)\}_{n \in \mathbb{Z}}$ and $E^{-}(\lambda) = \{E_n^{-}(\lambda)\}_{n \in \mathbb{Z}}$ are solutions of Eq. (2.1), and

$$F(\lambda) = W[E^{+}(\lambda), E^{-}(\lambda)]$$

holds. Using (4.2) and (4.5) we obtain that

$$\sigma_d(L) = \{\lambda : \lambda \in \Omega,\ F(\lambda) = 0\},$$
$$\sigma_{ss}(L) = \{\lambda : \lambda \in [-2,2],\ F(\lambda) = 0\}.$$

Now Theorem 4.2 yields the following

COROLLARY 4.1. *Under condition* (4.8) *the function F has a finite number of zeros in* Ω *and in* $[-2,2]$, *and each of them is of a finite multiplicity.*

Finally using (2.3), (2.4), and $\arccos \frac{\lambda}{2} = -i \ln((\lambda + \sqrt{\lambda^2 - 4})/2)$ we get that

$$E_n^+(\lambda) = \alpha_n^+ \left(\frac{\lambda + \sqrt{\lambda^2 - 4}}{2} \right)^n \times \left[1 + \sum_{m=1}^{\infty} A_{n,m}^+ \left(\frac{\lambda + \sqrt{\lambda^2 - 4}}{2} \right)^m \right], \qquad n \in \mathbb{Z},$$

$$E_n^-(\lambda) = \alpha_n^- \left(\frac{\lambda + \sqrt{\lambda^2 - 4}}{2} \right)^{-n} \times \left[1 + \sum_{\infty}^{m=-1} A_{n,m}^- \left(\frac{\lambda + \sqrt{\lambda^2 - 4}}{2} \right)^{-m} \right], \qquad n \in \mathbb{Z}.$$

5. PRINCIPAL VECTORS OF L

In this section we assume that (4.8) holds. Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ denote the zeros of the function F in Ω (which are the eigenvalues of L) with multiplicities $m_1, m_2, \ldots, m_p$, respectively. Similarly, let $\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_q$ be the zeros of F in $[-2,2]$ (which are the spectral singularities of L) with multiplicities $m_{p+1}, m_{p+2}, \ldots, m_q$, respectively. It is obvious that

$$\left\{ \frac{d^k}{d\lambda^k} W[E^+(\lambda), E^-(\lambda)] \right\}_{\lambda=\lambda_j} = \left\{ \frac{d^k}{d\lambda^k} F(\lambda) \right\}_{\lambda=\lambda_j} = 0, \tag{5.1}$$

for $k = 0, 1, \ldots, m_j - 1,\ j = 1, 2, \ldots, q.$

THEOREM 5.1. *The formula*

$$\left\{\frac{d^k}{d\lambda^k}E_n^+(\lambda)\right\}_{\lambda=\lambda_j} = \sum_{\nu=0}^{k}\binom{k}{\nu}\beta_{k-\nu}\left\{\frac{d^\nu}{d\lambda^\nu}E_n^-(\lambda)\right\}_{\lambda=\lambda_j}, \qquad n \in \mathbb{Z}, \tag{5.2}$$

for $k = 0, 1, \ldots, m_j - 1$, $j = 1, 2, \ldots, q$ *holds, where the constants* $\beta_0, \beta_1, \ldots, \beta_k$ *depend on* λ_j.

Proof. We will proceed by mathematical induction. Let $k = 0$. From (5.1) we get

$$E_n^+(\lambda_j) = \beta_0(\lambda_j)E_n^-(\lambda_j), \qquad n \in \mathbb{Z},$$

where $\beta_0(\lambda_j) \neq 0$. Let us assume that for $1 \leq k_0 \leq m_j - 2$, (5.2) holds; i.e.,

$$\left\{\frac{d^{k_0}}{d\lambda^{k_0}}E_n^+(\lambda)\right\}_{\lambda=\lambda_j} = \sum_{\nu=0}^{k_0}\binom{k_0}{\nu}\beta_{k_0-\nu}\left\{\frac{d^\nu}{d\lambda^\nu}E_n^-(\lambda)\right\}_{\lambda=\lambda_j}, \qquad n \in \mathbb{Z}. \tag{5.3}$$

Now we will prove that (5.2) holds for $k_0 + 1$. If $y(\lambda) = \{y_n(\lambda)\}_{n\in\mathbb{Z}}$ is a solution of (2.1), then $(d^k/d\lambda^k)y(\lambda) = \{(d^k/d\lambda^k)y_n(\lambda)\}_{n\in\mathbb{Z}}$ satisfies

$$a_{n-1}\frac{d^k}{d\lambda^k}y_{n-1}(\lambda) + b_n\frac{d^k}{d\lambda^k}y_n(\lambda) + a_n\frac{d^k}{d\lambda^k}y_{n+1}(\lambda) - \lambda\frac{d^k}{d\lambda^k}y_n(\lambda) = k\frac{d^{k-1}}{d\lambda^{k-1}}y_n(\lambda). \tag{5.4}$$

Writing (5.4) for $E^+(\lambda_j) = \{E_n^+(\lambda_j)\}_{n\in\mathbb{Z}}$ and $E^-(\lambda_j) = \{E_n^-(\lambda_j)\}_{n\in\mathbb{Z}}$ by putting $k = k_0$ and using (5.3) we find

$$a_{n-1}g_{n-1}(\lambda_j) + b_n g_n(\lambda_j) + a_n g_{n+1}(\lambda_j) - \lambda_j g_n(\lambda_j) = 0,$$

where $g(\lambda_j) = \{g_n(\lambda_j)\}_{n\in\mathbb{Z}}$ and

$$g_n(\lambda_j) = \left\{\frac{d^{k_0+1}}{d\lambda^{k_0+1}}E_n^+(\lambda)\right\}_{\lambda=\lambda_j} - \sum_{\nu=1}^{k_0+1}\binom{k_0+1}{\nu} \times \beta_{k_0+1-\nu}\left\{\frac{d^\nu}{d\lambda^\nu}E_n^-(\lambda)\right\}_{\lambda=\lambda_j}.$$

From (5.1) we have

$$W\left[g(\lambda_j), E^-(\lambda_j)\right] = \left\{\frac{d^{k_0+1}}{d\lambda^{k_0+1}} W[E^+(\lambda), E^-(\lambda)]\right\}_{\lambda=\lambda_j} = 0.$$

Hence there exists a constant $\beta_{k_0+1}(\lambda_j)$ such that

$$g_n(\lambda_j) = \beta_{k_0+1}(\lambda_j) E_n^-(\lambda_j), \qquad n \in \mathbb{Z}.$$

This shows that (5.2) holds for $k = k_0 + 1$. ∎

Using the notation

$$B_{k-\nu}(\lambda_j) = \frac{\beta_{k-\nu}(\lambda_j)}{(k-\nu)!},$$

we can write (5.2) as

$$\frac{1}{k!}\left\{\frac{d^k}{d\lambda^k} E_n^+(\lambda)\right\}_{\lambda=\lambda_j} = \sum_{\nu=0}^{k} B_{k-\nu}(\lambda_j) \frac{1}{\nu!}\left\{\frac{d^\nu}{d\lambda^\nu} E_n^-(\lambda)\right\}_{\lambda=\lambda_j}, \qquad n \in \mathbb{Z}, \tag{5.5}$$

$$k = 0, 1, \ldots, m_j - 1,\ j = 1, 2, \ldots, q.$$

DEFINITION 5.1. Let $\lambda = \lambda_0$ be an eigenvalue of L. If the vectors $y^{(0)}, y^{(1)}, \ldots, y^{(s)}$, $y^{(k)} = \{y_n^{(k)}\}_{n\in\mathbb{Z}}$, $k = 0, 1, \ldots, s$ satisfy the equations

$$(\ell y^{(0)})_n - \lambda_0 y_n^{(0)} = 0,$$

$$(\ell y^{(k)})_n - \lambda_0 y_n^{(k)} - y_n^{(k-1)} = 0, \qquad k = 1, 2, \ldots, s,\ n \in \mathbb{Z}, \tag{5.6}$$

then the vector $y^{(0)}$ is called the eigenvector corresponding to the eigenvalue $\lambda = \lambda_0$ of L. The vectors $y^{(1)}, \ldots, y^{(s)}$ are called the associated vectors corresponding to $\lambda = \lambda_0$. The eigenvector and the associated vectors corresponding to $\lambda = \lambda_0$ are called the principal vectors of the eigenvalue $\lambda = \lambda_0$.

The principal vectors of the spectral singularities of L are defined similarly.

Let us introduce the vectors

$$U^{(k)}(\lambda_j) := \left\{U_n^{(k)}(\lambda_j)\right\}_{n\in\mathbb{Z}}, \qquad k = 0, 1, \ldots, m_j - 1,\ j = 1, 2, \ldots, q, \tag{5.7}$$

where

$$U_n^{(k)}(\lambda_j) = \frac{1}{k!}\left\{\frac{d^k}{d\lambda^k}E_n^+(\lambda)\right\}_{\lambda=\lambda_j}, \qquad n \in \mathbb{Z} \tag{5.8}$$

or, by (5.5),

$$U_n^{(k)}(\lambda_j) = \sum_{\nu=0}^{k} B_{k-\nu}(\lambda_j)\frac{1}{\nu!}\left\{\frac{d^\nu}{d\lambda^\nu}E_n^-(\lambda)\right\}_{\lambda=\lambda_j}, \qquad n \in \mathbb{Z}. \tag{5.9}$$

From (5.4) and (5.7)–(5.9) we get that

$$\begin{gathered}\left(\ell U^{(0)}(\lambda_j)\right)_n - \lambda_j U_n^{(0)}(\lambda_j) = 0,\\ \left(\ell U^{(k)}(\lambda_j)\right)_n - \lambda_j U_n^{(k)}(\lambda_j) - U_n^{(k-1)}(\lambda_j) = 0,\\ k = 1,2,\ldots,m_j - 1, \qquad j = 1,2,\ldots,q.\end{gathered}$$

Consequently, the vectors $U^{(k)}(\lambda_j)$, $k = 0,1,\ldots,m_j - 1$, $j = 1,2,\ldots,p$, and $U^{(k)}(\lambda_j)$, $k = 0,1,\ldots,m_j - 1$, $j = p+1, p+2,\ldots,q$ are the principal vectors of eigenvalues and spectral singularities of L, respectively.

THEOREM 5.2.

$$\begin{aligned}&U^{(k)}(\lambda_j) \in \ell^2(\mathbb{Z}), \qquad k = 0,1,\ldots,m_j - 1,\ j = 1,2,\ldots,p\\ &U^{(k)}(\lambda_j) \notin \ell^2(\mathbb{Z}), \qquad k = 0,1,\ldots,m_j - 1,\ j = p+1,\ldots,q.\end{aligned}$$

Proof. Using $E_n^+(\lambda) = e_n^+(\arccos\frac{\lambda}{2})$ and $E_n^-(\lambda) = e_n^-(\arccos\frac{\lambda}{2})$ we obtain that

$$\left\{\frac{d^k}{d\lambda^k}E_n^+(\lambda)\right\}_{\lambda=\lambda_j} = \sum_{\nu=0}^{k} c_\nu^+\left\{\frac{d^\nu}{dz^\nu}e_n^+(z)\right\}_{z=z_j}, \qquad n \in \mathbb{Z}, \tag{5.10}$$

$$\left\{\frac{d^k}{d\lambda^k}E_n^-(\lambda)\right\}_{\lambda=\lambda_j} = \sum_{\nu=0}^{k} c_\nu^-\left\{\frac{d^\nu}{dz^\nu}e_n^-(z)\right\}_{z=z_j}, \qquad n \in \mathbb{Z}, \tag{5.11}$$

where $\lambda_j = 2\cos z_j$, $z_j \in P = P_0 \cup [-\frac{\pi}{2}, \frac{3\pi}{2}]$, $j = 1,2,\ldots,q$, and c_ν^+ and c_ν^- are constants depending on λ_j. From (2.3) and (2.4) we find that

$$\begin{aligned}&\left\{\frac{d^\nu}{dz^\nu}e_n^+(z)\right\}_{z=z_j}\\ &\qquad = \alpha_n^+ e^{inz_j}\left\{(in)^\nu + \sum_{m=1}^{\infty}[i(n+m)]^\nu A_{n,m}^+ e^{imz_j}\right\},\end{aligned} \tag{5.12}$$

(5.13)

$$\left\{\frac{d^{\nu}}{dz^{\nu}}e_n^-(z)\right\}_{z=z_j} = \alpha_n^- e^{-inz_j}\left\{(-in)^{\nu} + \sum_{-\infty}^{m=-1}[-i(n+m)]^{\nu}A_{n,m}^- e^{-imz_j}\right\},$$

$$n \in \mathbb{Z},\ j = 1, 2, \ldots, q.$$

For the principal vectors $U^{(k)}(\lambda_j) = \{U_n^{(k)}(\lambda_j)\}_{n \in \mathbb{Z}}$, $k = 0, 1, \ldots, m_j - 1$, $j = 1, 2, \ldots, p$ corresponding to the eigenvalues $\lambda_j = 2\cos z_j$, $j = 1, 2, \ldots, p$ of L we get, by (5.8) and (5.9), that

$$\sum_{n\in\mathbb{Z}} |U_n^{(k)}(\lambda_j)|^2 = \sum_{n=0}^{\infty}\left|\frac{1}{k!}\left\{\frac{d^k}{d\lambda^k}E_n^+(\lambda)\right\}_{\lambda=\lambda_j}\right|^2 + \sum_{-\infty}^{n=-1}\left|\sum_{\nu=0}^{k} B_{k-\nu}(\lambda_j)\frac{1}{\nu!}\left\{\frac{d^{\nu}}{d\lambda^{\nu}}E_n^-(\lambda)\right\}_{\lambda=\lambda_j}\right|^2. \tag{5.14}$$

Since $\operatorname{Im} z_j > 0$ for the eigenvalues $\lambda_j = 2\cos z_j$, $j = 1, 2, \ldots, p$ of L, (5.10) and (5.12) imply that the first series on the right hand side of (5.14) is convergent. Similarly (5.11) and (5.13) imply that the second series on the right hand side is also convergent. So we obtain

$$U^{(k)}(\lambda_j) \in \ell^2(\mathbb{Z}), \qquad k = 0, 1, \ldots, m_j - 1,\ j = 1, 2, \ldots, p.$$

If we consider (5.14) for the principal vectors corresponding to the spectral singularities $\lambda_j = 2\cos z_j$, $j = p+1, p+2, \ldots, q$ of L and consider that $\operatorname{Im} z_j = 0$ for the spectral singularities then we have, by (5.10)–(5.13), that

$$\sum_{n\in\mathbb{Z}} |U_n^{(k)}(\lambda_j)|^2 = \infty, \qquad k = 0, 1, \ldots, m_j - 1,\ j = p+1, p+2, \ldots, q.$$

Therefore we get

$$U^{(k)}(\lambda_j) \notin \ell^2(\mathbb{Z}), \qquad k = 0, 1, \ldots, m_j - 1,\ j = p+1, p+2, \ldots, q.$$

∎

Let us introduce Hilbert spaces

$$H_k(\mathbb{Z}) = \left\{y = \{y_n\}_{n\in\mathbb{Z}} : \sum_{n\in\mathbb{Z}}(1+|n|)^{2k}|y_n|^2 < \infty\right\}, \qquad k = 0, 1, \ldots,$$

$$H_{-k}(\mathbb{Z}) = \left\{u = \{u_n\}_{n\in\mathbb{Z}} : \sum_{n\in\mathbb{Z}}(1+|n|)^{-2k}|u_n|^2 < \infty\right\}, \qquad k = 0, 1, \ldots,$$

with

$$\|y\|_k^2 = \sum_{n\in\mathbb{Z}} (1+|n|)^{2k}|y_n|^2, \; \|u\|_{-k}^2 = \sum_{n\in\mathbb{Z}} (1+|n|)^{-2k}|u_n|^2,$$

respectively. It is obvious that $H_0(\mathbb{Z}) = \ell^2(\mathbb{Z})$ and

$$H_{k+1}(\mathbb{Z}) \subsetneqq H_k(\mathbb{Z}) \subsetneqq \ell^2(\mathbb{Z}) \subsetneqq H_{-k}(\mathbb{Z}) \subsetneqq H_{-(k+1)}(\mathbb{Z}), \quad k = 1, 2, \ldots . \tag{5.15}$$

Also $H_{-k}(\mathbb{Z})$ is isomorphic to the dual of $H_k(\mathbb{Z})$.

LEMMA 5.1. $U^{(k)}(\lambda_j) \in H_{-(k+1)}(\mathbb{Z})$, $k = 0, 1, \ldots, m_j - 1$, $j = p + 1, \ldots, q$.

Proof. Using the equation

$$\begin{aligned}\sum_{n\in\mathbb{Z}} &(1+|n|)^{-2(k+1)}|U^{(k)}(\lambda_j)|^2 \\ &= \sum_{n=0}^{\infty} (1+|n|)^{-2(k+1)}\left|\frac{1}{k!}\left\{\frac{d^k}{d\lambda^k}E_n^+(\lambda)\right\}_{\lambda=\lambda_j}\right|^2 \\ &\quad + \sum_{-\infty}^{n=-1} (1+|n|)^{-2(k+1)}\left|\sum_{\nu=0}^{k} B_{k-\nu}(\lambda_j)\frac{1}{\nu!}\left\{\frac{d^\nu}{d\lambda^\nu}E_n^-(\lambda)\right\}_{\lambda=\lambda_j}\right|^2,\end{aligned}$$

and considering (5.10)–(5.13) we get

$$\sum_{n\in\mathbb{Z}} (1+|n|)^{-2(k+1)}|U^{(k)}(\lambda_j)|^2 < \infty, \quad k = 0, 1, \ldots, m_j - 1, \; j = p+1, \ldots, q.$$

∎

Let

$$H_-(\mathbb{Z}) := H_{-m_0}(\mathbb{Z}),$$

where

$$m_0 = \max\{m_{p+1}, m_{p+2}, \ldots, m_q\}.$$

Now Lemma 5.1 and (5.15) yield the following immediately

THEOREM 5.3. $U^{(k)}(\lambda_j) \in H_-(\mathbb{Z})$, $k = 0, 1, \ldots, m_j - 1$, $j = p + 1, \ldots, q$.

6. DISCRETE DIRAC OPERATORS

By $\ell^2(\mathbb{Z}, \mathbb{C}^2)$ we denote the Hilbert space of all complex vector sequences

$$y = \left\{ \begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix} \right\}_{n \in \mathbb{Z}}$$

with the norm

$$\|y\|^2 = \sum_{n \in \mathbb{Z}} \left(|y_n^{(1)}|^2 + |y_n^{(2)}|^2 \right).$$

Let M denote the operator generated in $\ell^2(\mathbb{Z}, \mathbb{C}^2)$ by

$$(\Lambda y)_n = \begin{pmatrix} (\Lambda_1 y)_n \\ (\Lambda_2 y)_n \end{pmatrix} = \begin{pmatrix} \Delta y_n^{(2)} + p_n y_n^{(1)} \\ -\Delta y_{n-1}^{(1)} + q_n y_n^{(2)} \end{pmatrix}, \qquad n \in \mathbb{Z},$$

where $\{p_n\}_{n \in \mathbb{Z}}$ and $\{q_n\}_{n \in \mathbb{Z}}$ are complex sequences. In connection with the operator M we consider the system of difference equations

$$\left\{ \begin{array}{l} \Delta y_n^{(2)} + p_n y_n^{(1)} = \lambda y_n^{(1)} \\ -\Delta y_{n-1}^{(1)} + q_n y_n^{(2)} = \lambda y_n^{(2)} \end{array} \right., \qquad n \in \mathbb{Z}, \tag{6.1}$$

where λ is a spectral parameter.

If

$$\sum_{n \in \mathbb{Z}} |n| \left(|p_n| + |q_n| \right) < \infty \tag{6.2}$$

holds, then Eqs. (6.1) have unique solutions

$$\varphi_n(z) = \begin{pmatrix} \varphi_n^{(1)}(z) \\ \varphi_n^{(2)}(z) \end{pmatrix} = \left\{ E_2 + \sum_{m=1}^{\infty} \Phi_{n,m} e^{imz} \right\} \begin{pmatrix} e^{i\frac{z}{2}} \\ -i \end{pmatrix} e^{inz}, \qquad n \in \mathbb{Z},$$

$$\psi_n(z) = \begin{pmatrix} \psi_n^{(1)}(z) \\ \psi_n^{(2)}(z) \end{pmatrix} = \left\{ E_2 + \sum_{-\infty}^{m=-1} \Psi_{n,m} e^{-imz} \right\} \begin{pmatrix} -i \\ e^{i\frac{z}{2}} \end{pmatrix} e^{-inz}, \qquad n \in \mathbb{Z},$$

for $\lambda = 2 \sin \frac{z}{2}$ and $z \in \overline{\mathbb{C}}_+$, where

$$E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Phi_{n,m} = \begin{pmatrix} \Phi_{n,m}^{11} & \Phi_{n,m}^{12} \\ \Phi_{n,m}^{21} & \Phi_{n,m}^{22} \end{pmatrix}, \qquad \Psi_{n,m} = \begin{pmatrix} \Psi_{n,m}^{11} & \Psi_{n,m}^{12} \\ \Psi_{n,m}^{21} & \Psi_{n,m}^{22} \end{pmatrix}.$$

Moreover $\Phi^{ij}_{n,m}$ and $\Psi^{ij}_{n,m}$, $i, j = 1, 2$ are expressed in terms of $\{p_n\}_{n \in \mathbb{Z}}$ and $\{q_n\}_{n \in \mathbb{Z}}$. Also the solutions

$$\varphi(z) = \{\varphi_n(z)\}_{n \in \mathbb{Z}} = \left\{ \begin{pmatrix} \varphi_n^{(1)}(z) \\ \varphi_n^{(2)}(z) \end{pmatrix} \right\}_{n \in \mathbb{Z}}$$

and

$$\psi(z) = \{\psi_n(z)\}_{n \in \mathbb{Z}} = \left\{ \begin{pmatrix} \psi_n^{(1)}(z) \\ \psi_n^{(2)}(z) \end{pmatrix} \right\}_{n \in \mathbb{Z}}$$

are analytic with respect to z in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}}_+$ [5].

The Wronskian of two solutions

$$y = \left\{ \begin{pmatrix} y_n^{(1)}(\lambda) \\ y_n^{(2)}(\lambda) \end{pmatrix} \right\}_{n \in \mathbb{Z}} \qquad \text{and} \qquad u = \left\{ \begin{pmatrix} u_n^{(1)}(\lambda) \\ u_n^{(2)}(\lambda) \end{pmatrix} \right\}_{n \in \mathbb{Z}}$$

of (6.1) is defined by

$$W[y, u] = y_n^{(1)}(\lambda) u_{n+1}^{(2)}(\lambda) - y_{n+1}^{(2)}(\lambda) u_n^{(1)}(\lambda).$$

Let

$$a(z) = W[\varphi(z), \psi(z)], \qquad z \in \overline{\mathbb{C}}_+.$$

The function a is analytic in $\mathbb{C}_+$ and continuous in $\overline{\mathbb{C}}_+$, and

$$a(z + 4\pi) = a(z)$$

holds. It is clear that

$$\sigma_d(M) = \left\{ \lambda\colon\ \lambda = 2\sin\frac{z}{2},\ z \in \Pi,\ a(z) = 0 \right\},$$

$$\sigma_{ss}(M) = \left\{ \lambda\colon\ \lambda = 2\sin\frac{z}{2},\ z \in [0, 4\pi],\ a(z) = 0 \right\},$$

where

$$\Pi = \{ z\colon\ z = \xi + i\tau,\ 0 \le \xi \le 4\pi,\ \tau > 0 \}.$$

In a manner similar to Theorem 3.1, 4.1, and 4.2 we get

THEOREM 6.1. *Under condition* (6.2)

(i) $\sigma_c(M) = [-2, 2]$.

(ii) *The set of eigenvalues of M is bounded and countable, and its limit points lie only in* $[-2,2]$.

(iii) $\sigma_{ss}(M) \subset [-2,2]$, $\sigma_{ss}(M) = \overline{\sigma_{ss}(M)}$, *and* $\mu\{\sigma_{ss}(M)\} = 0$.

THEOREM 6.2. *If, for some* $\varepsilon > 0$,

$$\sup_{n \in \mathbb{Z}} \left\{ e^{\varepsilon|n|}(|p_n| + |q_n|) \right\} < \infty \tag{6.3}$$

holds, then the operator M has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.

Let us consider the functions

$$\Phi_n(\lambda) := \varphi_n\left(2\arcsin\frac{\lambda}{2}\right) = \begin{pmatrix} \varphi_n^{(1)}\left(2\arcsin\frac{\lambda}{2}\right) \\ \varphi_n^{(2)}\left(2\arcsin\frac{\lambda}{2}\right) \end{pmatrix} := \begin{pmatrix} \Phi_n^{(1)}(\lambda) \\ \Phi_n^{(2)}(\lambda) \end{pmatrix}, \qquad n \in \mathbb{Z},$$

$$\Psi_n(\lambda) := \psi_n\left(2\arcsin\frac{\lambda}{2}\right) = \begin{pmatrix} \psi_n^{(1)}\left(2\arcsin\frac{\lambda}{2}\right) \\ \psi_n^{(2)}\left(2\arcsin\frac{\lambda}{2}\right) \end{pmatrix} := \begin{pmatrix} \Psi_n^{(1)}(\lambda) \\ \Psi_n^{(1)}(\lambda) \end{pmatrix}, \qquad n \in \mathbb{Z},$$

$$A(\lambda) = a\left(2\arcsin\frac{\lambda}{2}\right).$$

The functions Φ_n, Ψ_n, $n \in \mathbb{Z}$, and A are analytic in $\Omega = \mathbb{C} \setminus [-2,2]$ and continuous up to the boundary of Ω. It is obvious that

$$\Phi(\lambda) = \{\Phi_n(\lambda)\}_{n \in \mathbb{Z}} = \left\{ \begin{pmatrix} \Phi_n^{(1)}(\lambda) \\ \Phi_n^{(2)}(\lambda) \end{pmatrix} \right\}_{n \in \mathbb{Z}}$$

and

$$\Psi(\lambda) = \{\Psi_n(\lambda)\}_{n \in \mathbb{Z}} = \left\{ \begin{pmatrix} \Psi_n^{(1)}(\lambda) \\ \Psi_n^{(2)}(\lambda) \end{pmatrix} \right\}_{n \in \mathbb{Z}}$$

are solutions of (6.1), and

$$A(\lambda) = W[\Phi(\lambda), \Psi(\lambda)].$$

Under condition (6.3) the function $A(\lambda)$ has a finite number of zeros in Ω and in $[-2,2]$, and each of them is of a finite multiplicity.

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ denote the zeros of the function A in Ω (which are the eigenvalues of M) with multiplicities $m_1, m_2, \ldots, m_k$, respectively. Similarly let $\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_\nu$ be the zeros of A in $[-2,2]$ (which are the spectral singularities of M) with multiplicities $m_{k+1}, m_{k+2}, \ldots, m_\nu$, respectively.

Analogously to (5.5) we get

$$\begin{pmatrix} \frac{1}{j!}\left\{\frac{d^j}{d\lambda^j}\Phi_n^{(1)}(\lambda)\right\}_{\lambda=\lambda_i} \\ \frac{1}{j!}\left\{\frac{d^j}{d\lambda^j}\Phi_n^{(2)}(\lambda)\right\}_{\lambda=\lambda_i} \end{pmatrix} = \begin{pmatrix} \sum_{\ell=0}^{j} F_{j-\ell}(\lambda_i)\frac{1}{\ell!}\left\{\frac{d^\ell}{d\lambda^\ell}\Psi_n^{(1)}(\lambda)\right\}_{\lambda=\lambda_i} \\ \sum_{\ell=0}^{j} F_{j-\ell}(\lambda_i)\frac{1}{\ell!}\left\{\frac{d^\ell}{d\lambda^\ell}\Psi_n^{(2)}(\lambda)\right\}_{\lambda=\lambda_i} \end{pmatrix}, \qquad n \in \mathbb{Z}, \tag{6.4}$$

$j = 0, 1, \ldots, m_i - 1$, $i = 1, 2, \ldots, k, k+1, \ldots, \nu$.

Let us introduce the vectors

$$V^{(j)}(\lambda_i) = \left\{V_n^{(j)}(\lambda_i)\right\}_{n\in\mathbb{Z}},$$
$$j = 0, 1, \ldots, m_i - 1,\ i = 1, 2, \ldots, k,\ k+1, \ldots, \nu,$$

where

$$V_n^{(j)}(\lambda_i) = \begin{pmatrix} \frac{1}{j!}\left\{\frac{d^j}{d\lambda^j}\Phi_n^{(1)}(\lambda)\right\}_{\lambda=\lambda_i} \\ \frac{1}{j!}\left\{\frac{d^j}{d\lambda^j}\Phi_n^{(2)}(\lambda)\right\}_{\lambda=\lambda_i} \end{pmatrix}$$
$$= \begin{pmatrix} \sum_{\ell=0}^{j} F_{j-\ell}(\lambda_i)\frac{1}{\ell!}\left\{\frac{d^\ell}{d\lambda^\ell}\Psi_n^{(1)}(\lambda)\right\}_{\lambda=\lambda_i} \\ \sum_{\ell=0}^{j} F_{j-\ell}(\lambda_i)\frac{1}{\ell!}\left\{\frac{d^\ell}{d\lambda^\ell}\Psi_n^{(2)}(\lambda)\right\}_{\lambda=\lambda_i} \end{pmatrix}, \qquad n \in \mathbb{Z},$$

by (6.4). The vectors $V^{(j)}(\lambda_i)$, $j = 0, 1, \ldots, m_i - 1$, $i = 1, 2, \ldots, k$, and $V^{(j)}(\lambda_i)$, $j = 0, 1, \ldots, m_i - 1$, $i = k+1, k+2, \ldots, \nu$ are the principal vectors of the eigenvalues and the spectral singularities of M, respectively.

Let $H_{-}(\mathbb{Z}, \mathbb{C}^2)$ denote the Hilbert space of vector sequences

$$y = \left\{\begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix}\right\}_{n\in\mathbb{Z}}$$

with the norm

$$\|y\|_-^2 = \sum_{n \in \mathbb{Z}} (1 + |n|)^{-2m_0} \left(|y_n^{(1)}|^2 + |y_n^{(2)}|^2 \right),$$

where

$$m_0 = \max\{m_{k+1}, m_{k+2}, \dots, m_\nu\}.$$

Analogously to Theorems 5.2 and 5.3 we have the following

THEOREM 6.3.

$$V^{(j)}(\lambda_i) \in \ell^2(\mathbb{Z}, \mathbb{C}^2), \qquad j = 0, 1, \dots, m_i - 1,\ i = 1, 2, \dots, k,$$

$$V^{(j)}(\lambda_i) \in H_-(\mathbb{Z}, \mathbb{C}^2), \qquad j = 0, 1, \dots, m_i - 1,\ i = k+1, k+2, \dots, \nu.$$

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